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Engaging English Learners With Equivalence as a Crosscutting Concept in Mathematics

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Equivalence is a concept that comes up across the entire K–12 grade span and in many different areas of mathematics. Students who are bureaucratically designated as "English Learners" can explore equivalence across their K–12 academic careers while developing language to connect ideas. Drawing on what is known generally from theory and research about equivalence, this brief identifies what educators can do to challenge and support English Learners in this area. Equivalence has the potential to be a crosscutting concept that connects ideas from algebra, geometry, and statistics.

The ideas explored in this brief emerged from field trials of a summer bridge program designed to challenge and support rising 9th grade students making the transition from middle to high school. The authors of this brief were part of the team that developed the bridge program, Reimagining and Amplifying Mathematics Participation, Understanding, and Practices (RAMPUP), through the National Research and Development Center to Improve Education for Secondary English Learners at WestEd. RAMPUP is a 3-week program with three modules: patterns, networks, and equivalence (Chu & Hamburger, 2022). Drawing from the RAMPUP field trials through which we iteratively developed the program, this brief gives examples of students' reasoning about and explorations of equivalence.

The brief is organized into the following sections:

- growing early understandings of "equals"
- stretching reasoning about proportionality with scaling
- generalizing to algebraic notions of equivalence
- developing diverse approaches to reasoning about equivalence
- examining examples from RAMPUP
- considering what educators can do to challenge and support English Learners

Growing Early Understandings of "Equals"

One of the foundational understandings in early mathematics is whether two arithmetic expressions are equal. The symbol for this relationship is the equals sign ("="). Often, young children have learned that the equals sign means "the answer is coming," as is typical of arithmetic exercises such as

$$2 + 3 =$$

As a consequence, sometimes students will do "chain arithmetic" in which they record operations serially, as in the following:

$$2 + 3 = 5 + 9 = 14$$

Here, the equals sign is being used to demonstrate that the answer to an operation is coming, but the three expressions are not all equal (Carpenter et al., 2003).

Students who conceptualize the equals sign in this way may encounter problems with arithmetic equations such as the following:

$$5 + 9 = \_\_\_ + 8$$

With an "answer coming" interpretation of the equals sign, students may say the number 14 belongs in the missing space.

Instead, it is more generative for students to develop an understanding of "=" as the relationship of equality between two arithmetic expressions. If they understand the equals sign as denoting that relationship, they are able to solve the arithmetic equation above ($5 + 9 = \_\_\_ + 8$) with multiple forms of reasoning, such as

- **having the same value:** "5 + 9 equals 14. And 6 + 8 also equals 14, so 5 + 9 = 6 + 8."
- **relating the two sides:** "I see the 9 and 8 are close. The 9 is bigger by 1, so the missing number on the other side is bigger by 1 as well. The missing number is 6."
- **transforming both sides:** "14 – 6 = 8 because if I add 6 to both sides, 14 = 8 + 6, which is true."

Stretching Reasoning About Proportionality With Scaling

Indeed, as students move into late elementary and middle grades, they will need to develop proportional reasoning that explains multiplicative relationships between quantities. From late elementary grades, students may begin with part–whole relationships associated with fractions, and in middle school they increasingly deal with pairs of quantities—such as distance and time—and geometric objects that have multiple dimensions (Ben-Chaim et al., 1998). Table 1 presents some ways that students may reason about proportionality across the late elementary to middle grades.

Table 1. Student Approaches to Fractions, Rates, and Similarity

	Fractions	Proportionality of rates	Similarity of figures
Having the same value	4/8 and 2/4 are equivalent because they are both equal to 1/2.	8 feet in 4 minutes and 4 feet in 2 minutes are equivalent because they are both equal to the unit rate 2 feet per minute.	A 4"x8" photo and a 2"x4" photo are equivalent because they have the same aspect ratio of 1:2.
Related to each other	4/8 and 2/4 are equivalent because 8 and 4 are twice 4 and 2, respectively.	6 feet in 3 minutes and 4 feet in 2 minutes are equivalent because the number of feet is in both cases twice the number of minutes.	These two figures have the same shape and same angles but different sizes.
Transformable	Multiplying 2/4 by 2/2 gives 4/8.	Scaling both distance and time by a factor of two transforms 4 feet in 2 minutes to 8 feet in 4 minutes.	Scaling by 2 both vertically and horizontally changes a 2"x4" into a 4"x8" photo.

Generalizing to Algebraic Notions of Equivalence

As students do more algebra in the middle grades, using the equals sign requires greater nuance because sometimes it is a statement about a particular value of an expression, such as

$$2x + 5 = 17$$

Here, there is a value for x that makes the expression $2x + 5$ have the same value as 17. By no means, however, are the expressions $2x + 5$ and 17 equivalent as polynomials—they only happen to be equal when $x = 6$.

Indeed, beginning in the middle grades, students may need to think about equivalence of expressions and equations in algebra in interesting ways, as shown in Table 2.

Table 2. Student Approaches to Equivalence of Expressions and Equations

	Expressions	Equations
Equivalent to the same thing	$2(x + 1) + 3$ and $2(x + 2) + 1$ are equivalent because when I simplify them, they have the same form $2x + 5$.	$2x + 5 = 17$ and $2x = 12$ have the same solution, $x = 6$.
Related to each other	$x + x + 5 = 2x + 5$ because $x + x = 2x$.	$x + x + 5 = 17$ is equivalent to $2x + 5 = 17$ because $x + x = 2x$.
Transformable	$2(x + 2) = 2x + 4$ because I can distribute the 2 out front.	$2x + 5 = 17$, $2x = 12$, and $x = 6$ are all equivalent because I can transform the first one by subtracting 5 from both sides. Then I take the result and divide both sides by 2.

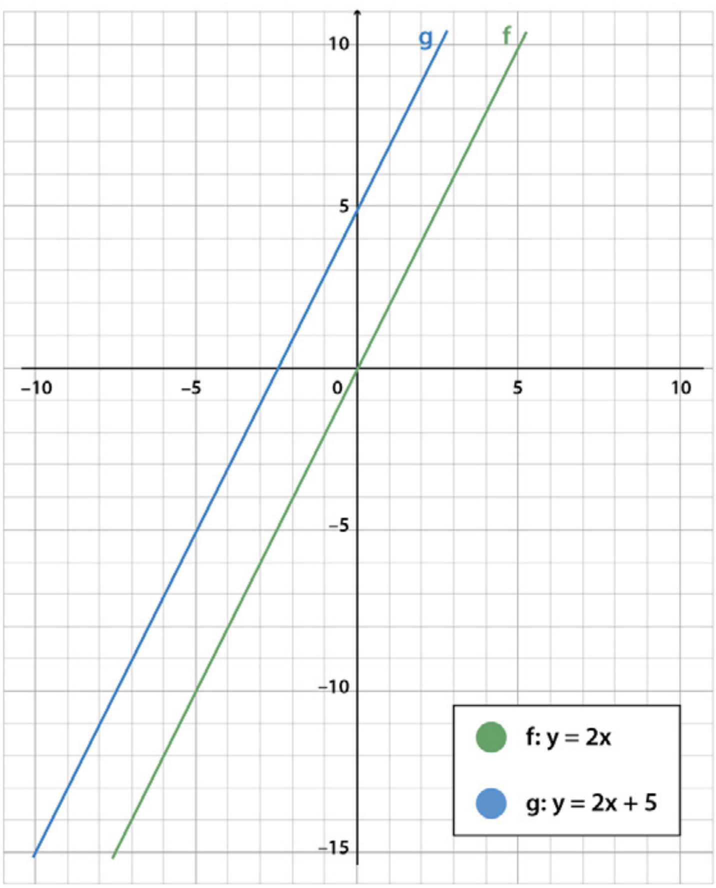
Success in middle grades mathematics has been associated with relational understandings of the equals sign (Knuth et al., 2006) and the ability to flexibly compare approaches that focus more on structure and relation than on mechanical multiplication or numerical evaluation of algebraic expressions (Rittle-Johnson & Star, 2009).

Developing Diverse Approaches to Reasoning About Equivalence

As students enter high school, they are expected to explore and connect concepts that may have multiple representations. This expansion is more complex but also enables more connections than the previous examples in which all objects were in the same representational family, such as in consideration of constants as algebraic expressions.

Consider what students might think and say when they are asked to compare what is the "same" or equivalent about the graphs of $y = 2x$ and $y = 2x + 5$ (Figure 1).

Figure 1. Graphs of Two Equations, $y = 2x$ and $y = 2x + 5$



- Students might say that the two graphs have the same slope by reading the equations and pointing at the parameter "2."
- Students might look at the graphs directly and trace how both graphs are moving in the same direction.
- Students may notice that vertically shifting one graph five units up gets the other graph.

These approaches align with a framework adapted from Cook and colleagues (2022) to describe approaches toward reasoning about the equivalence of two objects, such as linear graphs, as follows:

- **convergent:** Two objects are equivalent if they are related to a third object.[1] For instance, a student might note the same numerical quantity, such as rate of change in two equations, and reason that the graphs therefore have the same slope.[2]
- **relational:** Two objects are equivalent if they serve the same purpose or are related to each other in some way with regard to their characteristics. This approach may also be "descriptive" (Cook et al., 2022) in identifying attributes that connect (or distinguish) different objects. For instance, two graphs are equivalent because they are going in the same direction when compared visually.
- **transformational:** Two objects are equivalent if one can be transformed into another using some sequence of "allowable" moves. For instance, shifting the graph of $y = 2x$ up by 5 units transforms it into $y = 2x + 5$.

These are three general approaches that are well-suited to offering students multiple avenues of making mathematical connections and arguments.

Examining Examples From RAMPUP

This section offers examples from observations of RAMPUP field trials and analysis of written student work on a variety of examples in which equivalence was explored in contexts and concepts that overlap with but are not identical to typical school topics:

- Moving in Mazes
- Sliding Puzzles
- Graphs of Networks
- Mean as Point of Balance
- Functions as Alphabetic Codes
- Perspective Plane

1 Historically, the original postulate stated by Euclid was "things which are equal to the same, or to equals, are equal to each other" (Casey, 2022, p. 6).
2 Cook et al. (2002) referred to this as the "common characteristic" approach after Hamdan (2006) and Piaget (1977).

Moving in Mazes

One central metaphor for equivalence—"Can you get there from here?"—is introduced to students in the idea of mazes that they may be familiar with from early childhood, ones in which they complete "bird's-eye" puzzles that have walls they cannot jump over. Instead, they need to find a continuous path from one point to another of the maze. Indeed, in RAMPUP, students are asked to create their own mazes to analyze what are the connected components, which turn out to be equivalence classes.

Sliding Puzzles

Drawing upon an old mechanical puzzle of numbered wooden tiles in a square frame that can be slid around in order to achieve some configuration, in RAMPUP we have students explore whether they can get from one configuration to a targeted one. When one configuration can be transformed to the target configuration, they are equivalent.

Graphs of Networks

To contextualize the modeling of social and informational networks, in RAMPUP students explore graph theory, whereby they represent different ideas using graphs that have countable vertices, edges, and faces (Chu et al., under review). This topic is of great interest with regard to equivalence because it lends itself to both convergent approaches (any two connected graphs on the same surface will have the same Euler characteristics) and transformational approaches. Two graphs are equivalent if there is some sequence of transformations (additions or deletions) preserving connectedness that carry one into another.

Mean as Point of Balance

The most familiar concept from school mathematics that we address in RAMPUP is the average or arithmetic mean of a set of data. Unfortunately, students primarily think of the mean as a procedure rather than as a concept connected to real-world notions of balance (Hamburger & Chu, in press). We have students explore the mean as the point of balance of a dot plot with a focus on whether two dot plots have the same point of balance. The students employ a mix of approaches: transformational (moving data points without changing the overall point of balance), relational (looking at total distance to the left and right from a given point of balance), and convergent (calculating the mean by formula).

Functions as Alphabetic Codes

In the context of encrypting and decrypting messages, we have students in RAMPUP explore alphabetic codes with the English alphabet of 26 letters, assigning each letter a number (e.g., A = 1, B = 2, ... Z = 26). To enable them to carry out calculations when they obtain values larger than 26, we use the remainder when those values are divided by 26, also known as the integers modulo 26. For that reason, Z = 0, and students can explore what makes for “codes” that are decodable. For example, they soon discover that multiplying by 2 causes the alphabet to “loop” (i.e., A × 2 = 2 = B, but N × 2 = 28 = 2 = B as well), making the encoded message impossible to decode. The plaintext and encoded messages are equivalent if they have the same content. Students explore the operations of addition, multiplication, and exponentiation by looking at input/output graphs.

Perspective Plane

Coordinate geometry and linear functions in the Cartesian plane are essential topics in 8th grade mathematics (Lloyd et al., 2011). To offer RAMPUP students another way to look at relationships, we created an exploration of drawing in perspective. With the idea that when drawing in one-point perspective all lines $x = a$ would be parallel and vanish at a common point on the horizon, we have students explore how they can draw and coordinatize other points and lines in perspective when given $y = 0$, $y = 1$, and the lines $x = a$.

Table 3 provides examples of how we have heard students use convergent, relational, and transformational understandings of equivalence as they work in the contexts described above.

Table 3. Examples of Reasoning About Equivalence

	Convergent	Relational	Transformational
Concept	Two objects are equivalent if they are related to a third object.	Two objects are equivalent if they serve the same purpose or they are related to each other in some way.	Two objects are equivalent if one can be transformed into another using some sequence of “allowable” moves.
Moving in Mazes	These two points (A and B) are in the same region because I can get the same point C from there.	The regions are equivalent because both markers (e.g., spot A or spot B) are in the same region of the maze.	There is a path that connects spot A to spot B.
Sliding Puzzles	Two puzzles can both be rearranged to the “original” position.	Two puzzles are not equivalent because they have a “flipped” pair of tiles.	There is a sequence of moves (left/right, up/down) that allows me to get from the starting puzzle to the solved puzzle.

	Convergent	Relational	Transformational
Graphs of Networks	Two graphs have the same Euler characteristic.	One graph is a subgraph of another graph.	Graphs can be transformed into one another without changing connectedness.
Mean as Point of Balance	The data sets are equivalent because the mean, calculated by formula, is the same.	The distances to the left of the balance point are the same as the distances to the right of the balance point.	Points can be moved to the left and right until they all end up at the center.
Functions as Alphabetic Codes	The range of the function (code) is the full alphabet.	The graphs of the encryption function have similar shapes.	There is an inverse ("key of moves") that allows me to decipher the domain input.
Perspective Plane	Two lines meet at the same point on the horizon.	The rate of growth is the same on the grid in perspective.	Not applicable

Considering What Educators Can Do to Challenge and Support English Learners

The previous sections have all been about equivalence in general, as might be taken up by students across the K–12 grade span. This section focuses on three specific actions that educators can take specifically to support English Learners:

- ***Maintain the Challenge and Allow Language to Develop:*** English Learners are fully capable of engaging with these deep and generative ideas. Language is not a prerequisite for them to approach equivalence in multiple ways. Educators can build upon their uses of language to pull out these kinds of ideas (Walqui & van Lier, 2010) and can provide multiple opportunities for students to share their various ideas about equivalence. Please note that teachers do not need to share the categories of reasoning with their students, nor do students need to be able to classify their reasoning. Rather, educators should listen carefully to students' reasoning so that various perspectives can be discussed and honored in the classroom.
- ***Structure Activity to Highlight Relationships:*** As designed in RAMPUP, the activities through which students explored equivalence were all carefully designed in terms of grouping, structure, and language (Chu et al., 2023). Sorting tasks also have the potential to surface notions of groups and equivalence (Chu & Lopez, 2024). Educators should carefully consider the

structure of activities that elicit opportunities for students to share, describe, and respond to their developing notions of equivalence.

- ***Draw Attention to Approaches to Equivalence:*** It is not that students need to understand the convergent, relational, and transformational framework *per se*, but recognizing diverse approaches can be a way to fully value all student ideas. Educators can use how students talk about their ideas to connect to the framework, which may be through ideas such as "related to the same," "having the same shape," and "can move one into the other." The essence of students' ideas and the working definitions and language of the classroom are what matter.

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